Demonstration 1 (godel): An elementary study of the G"{o}del metric.

```plaintext
> restart:
> grtw();

GRTensorII Version 1.79 (R6)
2 February 2001
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Latest version available from: http://grtensor.phy.queensu.ca/
e:/Grtii(6)/Metrics

> grOptionTrace:=false:
> qload(godel1);

Default spacetime = godel1
For the godel1 spacetime:
Coordinates
x(up)
\[ x^a = [t, x, y, z] \]
Line element
\[ ds^2 = -dt^2 - 2e^{(\sqrt{2}\omega x)}\ dt \ dy + dx^2 - \frac{1}{2}e^{(2\sqrt{2}\omega x)}\ dy^2 + dz^2 \]

The Godel universe (c.f. Hawking and Ellis Section 5.7)

First we define a velocity field and then examine the kinematics

> grdef(`u{^a}:=\[1,0,0,0\]`);
Components assigned for metric: godel1
Created definition for u(up)

> grcalc(acc[u](up),expsc[u],shear[u](up,up),vor[u]);

Created a definition for u(up,cdn)
Created definition for shear(up,up)
Created definition for u(dn)
Created a definition for u(dn,cdn)
Created definition for acc(dn)
Created definition for vor(up,dn)

CPU Time = .419

> gralter(_,radical,expand,factor);
Component simplification of a GRTensorII object:

Applying routine `simplify[radical]` to object acc(up)[u]
Applying routine `simplify[radical]` to object expsc[u]
Applying routine `simplify[radical]` to object shear(up,up)[u]
Applying routine `simplify[radical]` to object vor[u]
Applying routine expand to object acc(up)[u]
Applying routine expand to object expsc[u]
Applying routine expand to object shear(up,up)[u]
Applying routine expand to object vor[u]
Applying routine factor to object acc(up)[u]
Applying routine factor to object expsc[u]
Applying routine factor to object shear(up,up)[u]
```
Applying routine factor to object \texttt{vor[u]}

\texttt{CPU Time = .080}

\texttt{> grdisplay(_);}  

\texttt{For the godell spacetime:}

\texttt{Acceleration vector}

\texttt{acc(up)}

\texttt{a^a = All components are zero}

\texttt{Expansion scalar}

\texttt{\Theta = 0}

\texttt{shear(up,up)}

\texttt{shear(up, up)}

\texttt{\sigma^a _b = All components are zero}

\texttt{Vorticity scalar}

\texttt{\omega^u = \sqrt{2 \cdot \text{\omega}^2}}

Einstein tensor is augmented with $\lambda$ (the cosmological constant).

\texttt{> grdef(`En{a b}:={G{a b}+\lambda*g{a b}`);}  

\texttt{Created definition for En(dn,dn)}

\texttt{> grcalc(En(up,up));}

\texttt{Created definition for En(up,up)}

\texttt{CPU Time = .040}

\texttt{> gralter(_,expand,factor);}  

Component simplification of a GRTensorII object:

Applying routine expand to object \texttt{En(up,up)}
Applying routine factor to object \texttt{En(up,up)}

\texttt{CPU Time = .010}

\texttt{> grdisplay(_);}  

\texttt{For the godell spacetime:}

\texttt{En(up,up)}

\texttt{En(up, up )}

\texttt{En^a _b =}

\begin{bmatrix}
3 \omega^2 + \lambda & 0 & -2 \frac{\omega^2 + \lambda}{\text{e}^{(\sqrt{2} \omega_0)}} & 0 \\
0 & \omega^2 + \lambda & 0 & 0 \\
-2 \frac{\omega^2 + \lambda}{\text{e}^{(\sqrt{2} \omega_0)}} & 0 & 2 \frac{\omega^2 + \lambda}{(\text{e}^{(\sqrt{2} \omega_0)})^2} & 0 \\
0 & 0 & 0 & \omega^2 + \lambda
\end{bmatrix}

It is clear that for dust

\texttt{> lambda:=-omega^2;}  

\lambda := -\omega^2
For the godel1 spacetime:

\[ E_{(up,up)}(up,up) \]

\[ E_{a\ b} = \begin{bmatrix} 2 \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

We now calculate the electric and magnetic components of the Weyl tensor associated with the velocity field \( u \).

\[ > \text{grcalc}(E[u](dn,dn),H[u](dn,dn)); \]

\[ \text{CPU Time} = .080 \]

\[ > \text{gralter}(_,6,7); \]

Component simplification of a GRTensorII object:

Applying routine expand to object \( E(dn,dn)[u] \)
Applying routine expand to object \( H(dn,dn)[u] \)
Applying routine factor to object \( E(dn,dn)[u] \)
Applying routine factor to object \( H(dn,dn)[u] \)

\[ \text{CPU Time} = .020 \]

\[ > \text{grdisplay}(_); \]

For the godel1 spacetime:

Electric part of Weyl

\[ E(dn,dn) \]

\[ E_{a\ b} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 \omega^2 & 0 & 0 \\ 0 & 0 & 1/6 \omega^2 \left( e^{\sqrt{2} \omega x} \right)^2 & 0 \\ 0 & 0 & 0 & -2/3 \omega^2 \end{bmatrix} \]

Magnetic part of Weyl

\[ H(dn,dn) \]

\[ H_{a\ b} = \text{All components are zero} \]

It is clear that the space has a high degree of symmetry. Here is a quick look for Killing vectors:

\[ > \text{KillingCoords}(); \]

Testing Killing coordinates for godel1
Created definition for coord1(dn)
Created a definition for coord1(dn,cdn)
Created a definition for coord1(up,cdn)
Created definition for coord2(dn)
Created a definition for coord2(dn,cdn)
Created a definition for coord2(up,cdn)
Created definition for coord3(dn)
Created a definition for coord3(\(dn,cdn\))
Created a definition for coord3(\(up,cdn\))
Created definition for coord4(\(dn\))
Created a definition for coord4(\(dn,cdn\))
Created a definition for coord4(\(up,cdn\))

CPU Time = 0.842

Killing Coordinate Test Results

Coordinate vector = \([t,x,y,z]\)

\(coord1(up) = [1, 0, 0, 0]\), a Killing vector.
\(coord2(up) = [0, 1, 0, 0]\), not a Killing vector.
\(coord3(up) = [0, 0, 1, 0]\), a Killing vector.
\(coord4(up) = [0, 0, 0, 1]\), a Killing vector.