Demonstration 1(kerr): Introduction to the Kerr metric, the vacuum ($\Lambda=0$) solution of Einstein’s equations for axial symmetry.

(kerr.mpl,newkerr.mpl,npdnkerr.mpl)

> restart;
> interface(labelling=false):
> grtw();

GRTensorII Version 1.79 (R6)
2 February 2001
Developed by Peter Musgrave, Denis Pollney and Kayll Lake
Copyright 1994-2001 by the authors.
Latest version available from: http://grtensor.phy.queensu.ca/
e:/Grtii(6)/Metrics

The most familiar form of the Kerr metric is in Boyer-Lindquist coordinates. We start by showing that the metric is indeed a vacuum solution (compare Landau and Lifshitz, p.323).

> qload(kerr);

Default spacetime = kerr
For the kerr spacetime:
Coordinates
x(up)

x' = [r, $\theta$, $\phi$, t]

Line element

\[ ds^2 = \frac{(r^2 + a^2 \cos(\theta)^2) \, dr^2}{r^2 - 2 \, m \, r + a^2} + (r^2 + a^2 \cos(\theta)^2) \, d\theta^2 \]

\[ + \sin(\theta)^2 \left( r^2 + a^2 + \frac{2 \, m \, r \, a^2 \sin(\theta)^2}{r^2 + a^2 \cos(\theta)^2} \right) \, d\phi^2 - \frac{4 \, m \, a \, r \, \sin(\theta)^2 \, d\phi \, dt}{r^2 + a^2 \cos(\theta)^2} \]

\[ + \left( -1 + \frac{2 \, m \, r}{r^2 + a^2 \cos(\theta)^2} \right) \, dt^2 \]

Kerr metric in Boyer-Lindquist coordinates.

> grcalc(R(dn,dn));

CPU Time = .300

> gralter(_,trig);
Component simplification of a GRTensorII object:
Applying routine `simplify[trig]` to object R(dn,dn)

CPU Time = .161

> grdisplay(_);

For the kerr spacetime:
Covariant Ricci
R(dn,dn)
\[ R_{a\ b} = \text{All components are zero} \]

The metric is flat for \( m = 0 \).

```plaintext
> grcalc(R(dn,dn,dn,dn));

CPU Time = .210
```

```plaintext
> grmap(_,subs,m=0,`x`);
Applying routine \( \text{subs} \) to \( R(dn,dn,dn,dn) \)
```

```plaintext
> gralter(_,trig);
Component simplification of a GRTensorII object:
Applying routine `\text{`simplify[trig]`} to object \( R(dn,dn,dn,dn) \)
```

```plaintext
CPU Time = .030
```

```plaintext
> grdisplay(_);
```

For the kerr spacetime:
Covariant Riemann
\[ R(dn,dn,dn,dn) = \text{All components are zero} \]

The time required to show that the metric is a vacuum solution depends very much on the coordinates.

For example, under the elementary transformation \( u = a \cos(\theta) \) we have:

```plaintext
> qload(newkerr);

Default spacetime = newkerr
```

For the newkerr spacetime:
Coordinates
\[ x^a = [r, u, \phi, t] \]

Line element
\[
 ds^2 = \frac{(r^2 + u^2) \, dr^2}{r^2 - 2 \, m \, r + a^2} + \frac{(r^2 + u^2) \, du^2}{a^2 - u^2} + \frac{(a^2 - u^2) \left( r^2 + a^2 + \frac{2 \, (a^2 - u^2) \, m \, r}{r^2 + u^2} \right) \, d\phi^2}{a^2} \\
- \frac{4 \, (a^2 - u^2) \, m \, r \, d\phi \, dt}{a \, (r^2 + u^2)} + \left( -1 + \frac{2 \, m \, r}{r^2 + u^2} \right) \, dt^2
\]

Constraints = \( [u = a \cos(\theta)] \)

The Kerr metric in Boyer-Lindquist type coordinates (\( u = a \cos(\theta) \)).

```plaintext
> grcalc(R(dn,dn));

CPU Time = .081
```

```plaintext
> grdisplay(_);
```

For the newkerr spacetime:
Covariant Ricci
\[ R(dn,dn) \]

\[ R_{a\ b} = \text{All components are zero} \]
The traditional way to argue that \( r=0 \) is singular only for \( \theta=\pi/2 \) is to calculate the Kretschmann scalar (e.g. Wald, p.315 Hawking and Ellis, p.162).

```maple
> grcalc(RiemSq);
Created definition for R(dn,dn,up,up)

CPU Time = .280
```

```maple
> gralter(_,6,7);
Component simplification of a GRTensorII object:

Applying routine expand to object RiemSq
Applying routine factor to object RiemSq

CPU Time = .040
```

We can put the scalar back into the original coordinates ($u=a*\cos(\theta)$).

```maple
> grmap(_,subs,u=a*cos(theta),`x`);
Applying routine subs to RiemSq

> grdisplay(_);

For the newkerr spacetime:

Full Contraction of Riemann

\[
K = 48 \, m^2 \left( -a \cos(\theta) + r \right) \left( r + a \cos(\theta) \right) \left( a^2 \cos(\theta)^2 - 4 \, r \, a \cos(\theta) + r^2 \right)
\]
\[
\left( a^2 \cos(\theta)^2 + 4 \, r \, a \cos(\theta) + r^2 \right) \left( r^2 + a^2 \cos(\theta)^2 \right)^6
\]

Since the solution is vacuum, the only non-vanishing curvature invariants are the Weyl invariants (e.g. Weinberg, p.146).

```maple
> grcalc(Winvars);
Scalar invariant library.
Created definition for C(up,up,up,up)

CPU Time = .471
```

```maple
> gralter(_,7);
Component simplification of a GRTensorII object:

Applying routine factor to object W1R
Applying routine factor to object W1I
Applying routine factor to object W2R
Applying routine factor to object W2I

CPU Time = .060
```

```maple
> grmap(_,subs,u=a*cos(theta),`x`);
Applying routine subs to W1R
Applying routine subs to W1I
Applying routine subs to W2R
Applying routine subs to W2I

> grmap(_,radsimp,`x`);
Applying routine radsimp to W1R
Applying routine radsimp to W1I
Applying routine radsimp to W2R
Applying routine radsimp to W2I

> gralter(_,7);
Component simplification of a GRTensorII object:
Applying routine factor to object W1R
Applying routine factor to object W1I
Applying routine factor to object W2R
Applying routine factor to object W2I

\[ \text{CPU Time} = .051 \]

\[ \text{?> grdisplay(_);} \]

For the newkerr spacetime:

\[ CM \text{ invariant } \text{Re}(W1) \]

\[ W1R = -6 \frac{m^2 (a \cos(\theta) - r) \left( r + a \cos(\theta) \right) \left( a^2 \cos(\theta)^2 - 4 r a \cos(\theta) + r^2 \right)}{(r^2 + a^2 \cos(\theta)^2)^6} \]

\[ CM \text{ invariant } \text{Im}(W1) \]

\[ W1I = -12 \frac{r m^2 a \cos(\theta) \left( -r^2 + 3 a^2 \cos(\theta)^2 \right) \left( -3 r^2 + a^2 \cos(\theta)^2 \right)}{(r^2 + a^2 \cos(\theta)^2)^6} \]

\[ CM \text{ invariant } \text{Re}(W2) \]

\[ W2R = -6 \frac{r m^3 \left( -r^2 + 3 a^2 \cos(\theta)^2 \right) \left( 3 a^6 \cos(\theta)^6 - 27 r^2 a^4 \cos(\theta)^4 + 33 r^4 a^2 \cos(\theta)^2 - r^6 \right)}{(r^2 + a^2 \cos(\theta)^2)^9} \]

\[ CM \text{ invariant } \text{Im}(W2) \]

\[ W2I = 6 \frac{a \cos(\theta) m^3 \left( -3 r^2 + a^2 \cos(\theta)^2 \right) \left( a^6 \cos(\theta)^6 - 33 r^2 a^4 \cos(\theta)^4 + 27 r^4 a^2 \cos(\theta)^2 - 3 r^6 \right)}{(r^2 + a^2 \cos(\theta)^2)^9} \]

The coordinates are obviously adapted to two Killing vectors (see further demonstrations under symmetry).

\[ \text{?> KillingCoords();} \]

Testing Killing coordinates for newkerr
Created definition for coord1(dn)
Created a definition for coord1(dn,cdn)
Created a definition for coord1(up,cdn)
Created definition for coord2(dn)
Created a definition for coord2(dn,cdn)
Created a definition for coord2(up,cdn)
Created definition for coord3(dn)
Created a definition for coord3(dn,cdn)
Created a definition for coord3(up,cdn)
Created definition for coord4(dn)
Created a definition for coord4(dn,cdn)
Created a definition for coord4(up,cdn)

\[ \text{CPU Time} = 1.412 \]

\[ \text{Killing Coordinate Test Results} \]

\[ \text{Coordinate vector} = \{ r, u, \phi, t \} \]
\[ \text{coord1}(up) = [1, 0, 0, 0], \text{ not a Killing vector.} \]
\[ \text{coord2}(up) = [0, 1, 0, 0], \text{ not a Killing vector.} \]
\[ \text{coord3}(up) = [0, 0, 1, 0], \text{ a Killing vector.} \]
We now show by way of the Frobenius theorem that the metric is not static unless \( a = 0 \) (e.g. Wald p.436):

\[
\text{coord4}(\up) = [0, 0, 0, 1], \text{ a Killing vector.}
\]

> \text{grdef(`xi{^a}:=\[0,0,0,1\}`)};
  Components assigned for metric: newkerr
  Created definition for \( \xi(\up) \)

> \text{grdef(`Xi{a b c}:=xi{[a ;c]}*xi{b}`)};
  Created definition for \( \xi(\dn) \)
  Created a definition for \( \xi(\dn,\cdn) \)
  Created definition for \( \Xi(\dn,\dn,\dn) \)

> \text{grcalc(Xi(\dn,\dn,\dn))};

\[
\text{CPU Time} = .030
\]

> \text{gralter(_,1)};
  Component simplification of a \text{GRTensorII} object:

\[
\text{Applying routine simplify to object \( \xi(\dn,\dn,\dn) \)}
\]

\[
\text{CPU Time} = .010
\]

> \text{grmap(_,subs,u=a*cos(theta),`x`)};
  Applying routine \text{subs} to \( \xi(\dn,\dn,\dn) \)

> \text{gralter(_,trig,factor)};
  Component simplification of a \text{GRTensorII} object:

\[
\text{Applying routine \`simplify[trig\` to object \( \xi(\dn,\dn,\dn) \)}
\]

\[
\text{Applying routine factor to object \( \xi(\dn,\dn,\dn) \)}
\]

\[
\text{CPU Time} = .040
\]

> \text{grcomponent(Xi(\dn,\dn,\dn),[r,phi,t])};

\[
-\frac{1}{3} a \frac{(-1+\cos(\theta)) (\cos(\theta)+1) m (a \cos(\theta) - r) (r + a \cos(\theta))}{(r^2 + a^2 \cos^2(\theta))^2}
\]

Consider now a covariant NPtetrad. We go directly to the Ricci and Weyl scalars:

> \text{qload(npdnkerr)};

\[
\text{Default spacetime} = \text{npdnkerr}
\]

\[
\text{For the npdnkerr spacetime:}
\]

\[
\text{Coordinates} \\
\quad x(\up) \\
\quad x^a = [t, r, u, \phi] \\
\]

\[
\text{Basis inner product} \\
\quad \eta(bup,bup) \\
\quad \eta^{(a)(b)} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

\[
\text{Null tetrad (covariant components)} \\
\quad \text{NPPl(\dn)}
\]
\[
l_a = \begin{bmatrix} 1, & -\frac{r^2 + u^2}{r^2 - 2Mr + a^2}, & 0, & -\frac{a^2 - u^2}{a} \end{bmatrix}
\]

NPn\(dn\)

\[
n_a = \begin{bmatrix} 1 \frac{r^2 - 2Mr + a^2}{r^2 + u^2}, & 1, & 0, & -\frac{1}{2} \frac{1}{a} \frac{(r^2 - 2Mr + a^2)(a^2 - u^2)}{a^2} \end{bmatrix}
\]

NPm\(dn\)

\[
m_a = \begin{bmatrix}
\frac{1}{2} \sqrt{a^2 - u^2} \frac{(ir + u)}{\sqrt{2}}, & \frac{1}{2} \frac{(r^2 + u^2)}{\sqrt{2}} & \frac{1}{2} \frac{1}{a} \frac{(ir + u + ir a^2 + ru a^2)}{\sqrt{2}} & \frac{1}{2} \frac{1}{a} \frac{a^2 - u^2}{\sqrt{2}}
\end{bmatrix}
\]

NPmbar\(dn\)

\[
m_{bar} = \begin{bmatrix}
\frac{1}{2} \sqrt{a^2 - u^2} \frac{(ir - u)}{\sqrt{2}}, & \frac{1}{2} \frac{(r^2 + u^2)}{\sqrt{2}}, & \frac{1}{2} \frac{1}{a} \frac{(ir - u + ir a^2 - ru a^2)}{\sqrt{2}} & \frac{1}{2} \frac{1}{a} \frac{a^2 - u^2}{\sqrt{2}}
\end{bmatrix}
\]

Covariant NPtetrad for Kerr metric \((u=a\cdot\cos(\theta))\) to Boyer-Lindquist coordinates

```plaintext
> grcalc(RicciSc, WeylSc):
'Basis/tetrad related object definitions'
'Last modified 23 January 2001'
Created a definition for e(bdn, dn, pdn)

CPU Time = .791
```

```plaintext
> gralter(_, 2, 7);
Component simplification of a GRTensorII object:
Applying routine `\'simplify[trig]\'` to object Phi100
Applying routine `\'simplify[trig]\'` to object Phi101
Applying routine `\'simplify[trig]\'` to object Phi102
Applying routine `\'simplify[trig]\'` to object Phi11
Applying routine `\'simplify[trig]\'` to object Phi12
Applying routine `\'simplify[trig]\'` to object Phi122
Applying routine `\'simplify[trig]\'` to object NPLambda
Applying routine `\'simplify[trig]\'` to object Psi10
Applying routine `\'simplify[trig]\'` to object Psi10
Applying routine `\'simplify[trig]\'` to object Psi12
Applying routine `\'simplify[trig]\'` to object Psi13
Applying routine `\'simplify[trig]\'` to object Psi14
Applying routine factor to object Phi100
Applying routine factor to object Phi101
Applying routine factor to object Phi102
Applying routine factor to object Phi11
Applying routine factor to object Phi12
```
Applying routine factor to object Phi22
Applying routine factor to object NPLambda
Applying routine factor to object Psi0
Applying routine factor to object Psi1
Applying routine factor to object Psi2
Applying routine factor to object Psi3
Applying routine factor to object Psi4

\[ CPU Time = .270 \]

\[ \text{grmap}(\_, \text{subs}, u=\text{a}\cos(\theta), \text{\`x`}); \]
Applying routine subs to Phi00
Applying routine subs to Phi01
Applying routine subs to Phi02
Applying routine subs to Phi11
Applying routine subs to Phi12
Applying routine subs to Phi22
Applying routine subs to NPLambda
Applying routine subs to Psi0
Applying routine subs to Psi1
Applying routine subs to Psi2
Applying routine subs to Psi3
Applying routine subs to Psi4

\[ \text{grdisplay}(\_); \]

For the npdnkerr spacetime:

Ricci Scalar, Phi00
\[ \Phi00 = 0 \]

Ricci Scalar, Phi01
\[ \Phi01 = 0 \]

Ricci Scalar, Phi02
\[ \Phi02 = 0 \]

Ricci Scalar, Phi11
\[ \Phi11 = 0 \]

Ricci Scalar, Phi12
\[ \Phi12 = 0 \]

Ricci Scalar, Phi22
\[ \Phi22 = 0 \]

\[ \text{NPLambda} := \frac{\text{Ricci Scalar}}{24} \]
\[ \text{NPLambda} = 0 \]

Weyl Scalar, NP Psi0
\[ \Psi0 = 0 \]

Weyl Scalar, NP Psi1
\[ \Psi1 = 0 \]

Weyl Scalar, NP Psi2
\[ \Psi2 = \frac{M}{(-r + I \text{a}\cos(\theta))^3} \]
A remarkable property of the metric is the existence of a Killing tensor (e.g. Wald p.321). See the demonstrations under symmetry.